Reflection and scattering of light from a three-layer structure with two rough interfaces

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The small-perturbation method is used to solve the boundary-value problem describing the scattering of light waves in a system of three media separated by rough surfaces. Explicit expressions are obtained for the average and fluctuating fields. The ellipsometric parameters $\Delta$ and $\Psi$ are constructed. The possibility of using these results to determine optically the statistical parameters of rough surfaces is discussed.

INTRODUCTION

The extensive use of layered structures and superlattices (including quantum superlattices) in microwave and optical electronics, and also the trend toward microminiaturization significantly increase the requirements on the quality of the geometric characteristics of interfaces between media. Indeed, for complex multilayer elements with characteristic linear dimensions of order 100 Å, interfacial roughness significantly affects the electrophysical and optical properties of the device.

It is well known that the roughness of a surface can vary widely as a result of etching, oxidation, epitaxy, and so on. Hence it is necessary to be able to quickly and accurately determine the statistical parameters of rough surfaces such as the standard deviation $\sigma$ of the heights of the irregularities and the correlation length $L$.

There are several methods of determining the geometrical characteristics of surfaces. Optical methods are the most promising in our opinion, because they allow nondestructive testing. An optical method which is relatively simple and highly sensitive is the ellipsometric method, in which the phase relations between reflected (scattered) light waves of different polarizations are analyzed. An ellipsometric measuring system, together with computer programs to analyze the experimental data, provides a unique capability of monitoring technological processes in real time. However the results obtained will be relevant to the real situation only when the effect of the interfacial roughness on the reflection (scattering) of light is taken into account.

The interaction of different types of waves with rough surfaces occurs in many fields of physics such as acoustics, radiophysics, optics, molecular physics, and so on. In spite of its more than 300 year history, the problem is far from a final solution. The methods used to attack this problem can be classified into three groups: the small-perturbation method, Kirchhoff's method, and the two-scale model. The small-perturbation method is the best justified mathematically. It is based on expansion of the reflected (scattered) waves in a power series in the two parameters $\omega L << 1$ and $\omega \sigma << 1$, where $\omega$ is the wave vector. The small-perturbation method has been most fully worked out in Refs. 2-4, where the formal scattering problem was solved using perturbation theory and rules for the calculation of the terms of the Born series to arbitrary finite order were given. In the opposite limiting case, when $\min(\omega, L)^{-k} >> 1$, one uses the Kirchhoff method, which is the quasiclassical approximation for the scattering problem. The two-scale model is used when the surfaces have large-scale irregularities modulated by small "ripples".

Previous papers have considered the scattering of light on a single rough surface. This assumption significantly limits the applicability of the results, since it describes the interaction of light only in the case of very pure metallic or large, strongly absorbing objects, where the effect of the back surface can be neglected. But thin organic or oxide films form naturally on the surfaces of materials, and therefore any scattering object has at least two rough surfaces which make comparable contributions to the photoresponse. Multilayer structures, such as insulator—semiconductor structures, which are essential elements in charge-coupled devices and field-effect transistors, and metal—insulator—semiconductor structures are also of great interest.

In the present paper we consider the interaction of light waves with two statistically independent and statistically uniform rough surfaces. Explicit expressions are obtained for the ellipsometric parameters for an isotropic autocorrelation function of arbitrary form. For appropriate values of the parameters the results go over to the known expressions for a single surface. The dependence of the ellipsometric parameters on the angle of incidence $\Theta$ and on the standard deviation $\sigma$ of the heights and the correlation length $L$ is studied (Figs. 2-8). As a comparison with previous results, calculations are done for a Gaussian autocorrelation function in the case of a single surface. The practical application of our method to the determination of surface roughness and its extension to statistically anisotropic and correlated interfaces are also discussed.

LIGHT SCATTERING BY ROUGH SURFACES

We consider the scattering of a plane monochromatic light wave of frequency $\omega$ incident at an angle $\Theta$ on a system (Fig. 1) of three media with complex dielectric constants $\varepsilon_{nm}$ ($m = 1, 2, 3$). The surfaces between the media are on average parallel to one another and separated by a distance $a$. The
FIG. 1. Scattering of light waves by an object with two rough surfaces: \( \sigma \) is the standard deviation of the heights of the irregularities, \( L \) is the correlation length \((i = 1, 2)\), \( \varepsilon_m \) is the dielectric constant of the medium \((m = 1, 2, 3)\), \( \Theta \) is the angle of incidence of the light wave, \( d \) is the effective thickness of the second medium.

surfaces are described by the random normal functions \( f_n(x) \). It is well known\(^6\) that \( f_n(x) \) is completely described by its average value \( \langle f_n(x) \rangle \), its variance \( \sigma_n^2 = \langle f_n^2(x) \rangle \), and its autocorrelation function \( W_n(x_1,x_2) = \sigma_n^{-2} \langle f_n(x_1)f_n(x_2) \rangle \). Here and below the angle brackets denote a statistical average of a random quantity with respect to the appropriate probability density.

To calculate the electromagnetic fields in the system it is necessary to solve a boundary-value problem. We assume that

\[
\max(\sigma_n/\varepsilon_m)^{1/2} \omega/c \ll 1, \quad \max(\sigma_n/L_n) \ll 1, \quad (1)
\]

where \( c \) is the speed of light in vacuum and \( L_n \) is the smallest of the correlation lengths for the autocorrelation function. It follows from (1) that we can use the small-perturbation method. The electromagnetic fields \( E_m, H_m \) in the \( m \)-th medium are written as sums of average \( E_m, H_m \) and fluctuating \( e_m, h_m \) components\(^1,2\).}

\[
E_m = E_m + e_m, \quad H_m = H_m + h_m, \quad (2)
\]

where, in view of the assumed statistical uniformity of the surfaces, the average fields \( E_m, H_m \) obey Snell's law and have the same form as the fields in the case of plane bound-
aries. Unlike the latter case, however, here the reflection \( R_m \) and transmission \( T_m \) coefficients of the average field are not given by the simple Fresnel formulas\(^7\) but are unknown quantities to be determined as part of the solution of the boundary-value problem.

Using the representation (2), the linearity of Maxwell's equations, and the above inequalities, we expand the boundary conditions in the small parameters and obtain a closed system of equations for the reflection and transmission coefficients of the average fields and the amplitudes of the fluctuating fields. In the coordinate system defined with respect to the plane of incidence (Fig. 1) these equations have the following form to the lowest nonvanishing order in the small parameters, assuming that the first and second surfaces are statistically independent:

\[
\begin{align*}
\left\{ \hat{E}_{i-1}(z) [E_m(x,z) - E_{i-1,m}(x,z)] 
+ f_{i-1}(x) \frac{\partial}{\partial z} (e_m(x,z) - e_{i-1,m}(x,z)) \right\}
&= 0, \\
\left\{ \hat{H}_{i-1}(z) [H_m(x,z) - H_{i-1,m}(x,z)] 
+ f_{i-1}(x) \frac{\partial}{\partial z} (h_m(x,z) 
- h_{i-1,m}(x,z)) \right\}
&= 0
\end{align*}
\]

and

\[
\begin{align*}
\left\{ e_m(x,z) - e_{i-1,m}(x,z) + A_{i-1,m}(x,z) \right\}
&= 0, \\
\left\{ h_m(x,z) - h_{i-1,m}(x,z) + B_{i-1,m}(x,z) \right\}
&= 0
\end{align*}
\]

where

\[
\hat{E}_i(z) = 1 + \frac{\sigma_i^2}{2} \frac{\partial^2}{\partial z^2}, \quad \gamma_{i,x}(x) = \frac{\partial f_i(x)}{\partial x}, \quad \gamma_{i,y}(x) = \frac{\partial f_i(x)}{\partial y}, \quad z_2 - z_1 = d, \quad \alpha = x, y; \quad i = 2, 1; \quad m = 1, 2, 3;
\]

and the functions \( A_{i,m}(x, z) \) and \( B_{i,m}(x, z) \) are:

\[
A_{i,m}(x, z) = f_i(x) \frac{\partial}{\partial x} [E_{i-1,m}(x, z) - E_{i,m}(x, z)] + \gamma_{i,x}(x) \times (E_{i-1,m}(x, z) - E_{i,m}(x, z))
\]

\[
B_{i,m}(x, z) = f_i(x) \frac{\partial}{\partial x} [H_{i-1,m}(x, z) - H_{i,m}(x, z)].
\]

FIG. 2. Dependence of the ellipsometric parameters \( \Delta, \Psi \) on the angle of incidence \( \Theta \) for an object with one surface: \( \sigma_1 = 0.1, \ell_1 = 1.0, \varepsilon_1 = 1.0 - 0.0i \text{ (air)}, \ell_2 = 15.26 - 0.704i (Si), \ell_{mz} = 1.95 eV. \)
The reflection and transmission coefficients of the average fields and the amplitudes of the fluctuating fields are determined from (3) and (4) as follows. First the system of equation (4) is solved, treating \( A_{\pm,\alpha}(x, z) \) and \( B_{\pm,\alpha}(x, z) \) as known quantities. This is conveniently done in the wave-vector representation, which is related to the coordinate representation by the two-dimensional Fourier transform

\[
e_{1}(x, z) = \int_{-\infty}^{+\infty} \frac{dq}{(2\pi)^2} e_{1}(q) \exp[-i\beta z + iqx],
\]

\[
e_{2}(x, z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} dq \{ e_{+}^{2}(q)e^{i\beta z} + e_{-}^{2}(q)e^{i\beta z} \} \exp[iqx],
\]

\[
e_{1}(x, z) = \int_{-\infty}^{+\infty} \frac{dq}{(2\pi)^2} e_{1}(q) e^{i\beta z + iqx},
\]

\[
A_{\pm,\alpha}(x, z) = \int_{-\infty}^{+\infty} \frac{dq}{(2\pi)^2} A_{\pm,\alpha}(q)e^{iqx},
\]

\[
B_{\pm,\alpha}(x, z) = \int_{-\infty}^{+\infty} \frac{dq}{(2\pi)^2} B_{\pm,\alpha}(q)e^{iqx}.
\]

In writing down the fluctuating fields in the first and third media we have used the Rayleigh hypothesis. As was shown in Ref. 8, this is a good approximation if \( \gamma_{1,\alpha}(x) = \sigma_{2}/L_{n} \leq 0.4 \). In our case this inequality is satisfied, since \( \sigma_{2}/L_{n} \) is a small parameter of the theory [Eq. (1)]. The solution of (4) in the wave-vector representation is conveniently written in the form

\[
e_{1}(q) = \sum_{i,\alpha} [C_{1,\pm,\alpha}(q) + D_{1,\pm,\alpha}(q)],
\]

\[
e_{2}(q) = \sum_{i,\alpha} [C_{2,\pm,\alpha}(q) + D_{2,\pm,\alpha}(q)],
\]

\[
e_{3}(q) = \sum_{i,\alpha} [C_{3,\pm,\alpha}(q) + D_{3,\pm,\alpha}(q)],
\]

\[
i = 1, 2; \quad \alpha = x, y, z; \quad \beta = x, y, z.
\]

The coefficients \( C_{i,\pm,\alpha}(q) \) and \( D_{i,\pm,\alpha}(q) \) are given in Appendix A. We next substitute expressions (6) and (7) for the fluctuating fields into system (3), which becomes closed by virtue of (5). The equations were solved for \( s \)- and \( p \)-polarized incident light waves. The results in the general case are quite complicated, and we therefore assume statistically isotropic surfaces and present the explicit expressions only for the average fields \( E_{1}^{(p)}, H_{1}^{(p)} \), in the first medium, since they are needed to construct the ellipsometric parameters \( \Psi \) and \( \Delta \). Because of the statistical isotropy of the surface, for an \( s \)-polarized incident wave the nonzero components of the average fields in the first medium are

\[
E_{1}^{i}(x, z) = R_{1,1}E_{0}^{i}\exp[ik_{1}z - ik_{1}z],
\]

\[
H_{1}^{i}(x, z) = E_{1}^{i}(x, z) \cos \Theta, \quad H_{1}^{s}(x, z) = E_{1}^{i}(x, z) \sin \Theta.
\]

and for a \( p \)-polarized wave the nonzero components are

\[
E_{1}^{p}(x, z) = R_{1,1}H_{1}^{p}\exp[ik_{1}z - ik_{1}z],
\]

\[
E_{1}^{s}(x, z) = -\cos \Theta H_{1}^{p}(x, z), \quad k_{1}z = k_{0} \sin \Theta,
\]

\[
E_{1}^{s}(x, z) = -H_{1}^{p}(x, z) \sin \Theta, \quad k_{1}z = (k_{1}^{2} - k_{2}^{2})^{1/2}, \quad k_{0} = \omega/c.
\]

where \( E_{0}^{i} \) and \( H_{1}^{p} \) are the amplitude of the incident waves of polarizations \( s \) and \( p \) and \( R_{1,1}, R_{1,1} \), are the reflection coefficients for the average fields.

The ellipsometric parameters \( \Psi \) and \( \Delta \) can be written in terms of \( R_{1,1}, R_{1,1} \) in the usual way

\[
\exp[\Delta i \text{eg} \Psi] = R_{1,1}. \quad R_{1,1}.
\]

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The reflection coefficients $R_{np}$ have the form

$$R_{12} = \frac{R_{12}^o + R_{23}^o T_{12}^o T_{23}^o - R_{23}^o R_{12}^o \exp(2ik_{2d}d)}{1 - R_{23}^o R_{12}^o \exp(2ik_{2d}d)}, \quad \alpha = s, p$$ (11)

because of the multiple-wave interference effect. Because of the surface roughness, the reflection coefficients $R_{np}$ of a single surface are not antisymmetric under permutation of the indices $n$ and $m$:

$$R_{12} = r_{12}^o (1 - 2\sigma^2 k_{12}^o + M_{12}),$$
$$R_{12} = r_{12}^o (1 - 2\sigma^2 k_{12}^o + M_{12}),$$
$$R_{12} = r_{12}^o (1 - 2\sigma^2 k_{12}^o + M_{12}),$$

$$T_{12}^o T_{23}^o = \left[1 - 2\sigma^2 (k_{12}^o + M_{12})\right],$$

$$R_{12}^p = r_{12}^p (1 - 2\sigma^2 s_{12} k_{12}^o),$$
$$R_{12}^p = r_{12}^p (1 - 2\sigma^2 s_{12} k_{12}^o),$$
$$T_{12}^p T_{23}^p = 1 + 2\sigma^2 s_{12} k_{12}^o$$

In (12) we have introduced the notation

$$s_{nm} = \frac{k_{nm}^2 k_{nm}^2 (k_{nm}^2 - k_{nm}^2)}{k_{nm}^2 k_{nm}^2 + k_{nm}^2 k_{nm}^2}, \quad t_{nm} = \frac{k_{nm} - k_{nm}}{k_{nm} + k_{nm}},$$

$$r_{nm} = \frac{k_{nm} - k_{nm}}{k_{nm} + k_{nm}},$$

$$s_{nm} = \frac{k_{nm}^2 k_{nm}^2 (k_{nm}^2 - k_{nm}^2)}{k_{nm}^2 k_{nm}^2 + k_{nm}^2 k_{nm}^2}, \quad t_{nm} = \frac{k_{nm} - k_{nm}}{k_{nm} + k_{nm}},$$

$$r_{nm} = \frac{k_{nm} - k_{nm}}{k_{nm} + k_{nm}},$$

$$s_{nm} = \frac{k_{nm}^2 k_{nm}^2 (k_{nm}^2 - k_{nm}^2)}{k_{nm}^2 k_{nm}^2 + k_{nm}^2 k_{nm}^2}, \quad t_{nm} = \frac{k_{nm} - k_{nm}}{k_{nm} + k_{nm}},$$

$$r_{nm} = \frac{k_{nm} - k_{nm}}{k_{nm} + k_{nm}},$$

The quantities $M_{np}$, $N_p$, $P_p (n = 1, 2; \alpha = x, y)$ are two-dimensional integrals of the form

$$\int_{-\infty}^{+\infty} \frac{dq}{(2\pi)^2} W_n(q_x - k_{1x}, q_y) \{M_{no}(q), N_o(q), P_o(q)\},$$ (14)

where $W_n(q_x - k_{1x}, q_y)$ is the Fourier transform of the autocorrelation function of the first $(n = 1)$ or second $(n = 2)$ surface, and explicit expressions for the functions $M_{no}(q)$, $N_{o}(q)$, $P_{o}(q)$ are given in Appendix B.

It follows from (11) that when the variances of the heights of the first and second interfaces are set equal to zero, the reflection coefficients $R_{np}$ reduce to the results (see Ref. 7, for example) for a single interfacial surface from (11). For $\varepsilon_3 = 0$, Eq. (11) describes the reflection of light from the boundary between the first and second media, and for $\varepsilon_1 = 0$, Eq. (11) describes the reflection of light from the boundary between the second and third media. As in the absence of roughness, the reflection coefficients in the second case have the additional phase factor $\exp(2ik_{1d}d)$ because the boundary lies a distance $d$ from the origin of the coordinate system. The third way of obtaining the reflection coefficients for a single surface from (11) is to set the thickness of the second medium $d$ equal to zero and to equate the variances of the heights and autocorrelation functions of the first and second surfaces. Then Eq. (11) will describe the reflection of light from the boundary between the first and second media.

The reflection coefficients for a single surface obtained by the above methods agree with the known results (see Ref. 3 for example) if they are written in the same notation.

**DISCUSSION OF THE RESULTS**

To proceed further the explicit form of the autocorrelation function must be specified. We assume that the autocorrelation functions of the first and second interfacial surfaces...
are single-parameter Gaussians of the form
\[ W_n(q_x - k_{1x}, q_y) = \pi L_n^2 \exp \left\{ -L_n^2 \left\{ (q_x - k_{1x})^2 + q_y^2 \right\}/4 \right\}; \quad n = 1, 2. \]  
(15)

This form is chosen because it is often used in models of randomly rough surfaces in both theoretical calculations and in the interpretation of experimental data.\(^9\) In addition, when (15) is put into (14) of the integrations can be performed explicitly, which greatly simplifies the numerical calculation of the reflection coefficients and the ellipsometric parameters.

In the numerical calculations the parameters \( \sigma \) and \( L \) are measured in units of \( k_0 = \text{calc} \). Figures 2-8 show the numerical results for the ellipsometric parameters \( \Delta, \Psi \) and the changes in these parameters \( \delta \Delta = \Delta - \Delta_0, \delta \Psi = \Psi - \Psi_0 \) \((\Delta_0, \Psi_0 \) are the ellipsometric parameters in the absence of roughness) as functions of the angle of incidence \( \Theta \), the standard deviation \( \sigma \) of the heights, and the correlation length \( L \) for a single rough surface \( \varepsilon_2 = \varepsilon_3 \). A single surface was assumed so that our results could be compared to previous results\(^2-4\) and also to more easily demonstrate the main qualitative features of the reflection (scattering) of light from a rough surface.

The changes in the ellipsometric parameters \( \delta \Delta, \delta \Psi \) due to the presence of surface roughness are shown in Figs. 3-6. As would be expected, \( \delta \Delta \) and \( \delta \Psi \) vary most strongly with \( \Theta \) near the effective Brewster angle.\(^7\) The absolute values of \( \delta \Delta \), \( \delta \Psi \) decrease with increasing correlation length \( L \) (Figs. 3 and 4) or with decreasing standard deviation \( \sigma \) of the heights (Figs. 5 and 6). The dependence of \( \delta \Delta, \delta \Psi \) on \( \sigma \) is obvious and the dependence on \( L \) is connected with the fact that, in the limit \( L \to \infty \), Eq. (12) becomes

\[ R_{12}^{\alpha \beta} = \tau_{12}^{\alpha \beta} (1 - \sigma^2 k_{1x}^2), \]

and therefore from (10) the ellipsometric parameters \( \Delta \) and \( \Psi \) do not depend on the roughness parameters. This is seen more clearly in Figs. 7 and 8, which show the dependence of \( \delta \Delta, \delta \Psi \) on the correlation length \( L \).

### CONCLUSION

Let us summarize the main results of this paper. Light scattering from a pair of rough surfaces has been considered using the small-perturbation method. Expressions have been obtained for the average and fluctuating fields and also for the reflection and transmission coefficients for \( \alpha \) and \( \beta \)-polarized incident waves as functions of the roughness parameters, the effective thickness of the film, and the angle of incidence of the light for complex dielectric constants. Various physical quantities such as the scattering indicatrix and the Stokes parameters can be obtained from these expressions. In particular, we calculated the ellipsometric parameters \( \Delta \) and \( \Psi \), which can be used to solve the inverse scattering problem\(^10,11\) to determine the statistical parameters of the surface irregularities. A computer program has been written to automate the ellipsometric system for this task.

A generalization of the approach considered here is to solve the direct scattering problem in the case of a large number of rough interfacial surfaces with possible correlations between the irregularities on different surfaces. Another generalization is to remove the restriction \( r k \ll 1 \). It can be shown\(^12\) that the only necessary condition for the small-perturbation method is \( r L \ll 1 \). Therefore one can attempt to solve the direct scattering problem for arbitrary values of the ratio of the irregularity heights to the irregularity heights to the wavelength. The solution to this problem would reduce to the solutions obtained by Kirchhoff's method and the usual small-perturbation method in the appropriate limiting cases.

### APPENDIX A

The coefficients determining the fluctuating fields \((7)\) are

\[ C_{1\gamma, i\alpha} = \delta_{i\alpha} \beta_{\gamma, i\alpha} + C_{2\gamma, i\alpha} + C_{3\gamma, i\alpha}; \]

\[ D_{1\gamma, i\alpha} = D_{2\gamma, i\alpha} + D_{3\gamma, i\alpha}; \]

\[ \beta_{1} C_{1z, i\alpha} = q_{z} C_{1z, i\alpha} + q_{y} C_{1y, i\alpha}; \]

\[ \beta_{1} D_{1z, i\alpha} = q_{x} D_{1z, i\alpha} + q_{y} D_{1y, i\alpha}; \]

\[ C_{3z, i\alpha} = \left[ U_{1}^{-1}(q) \left( \tau_{21}^{y2y1} + \tau_{21}^{r2y1} \right) \right]; \]

\[ D_{3z, i\alpha} = \left[ U_{1}^{-1}(q) \left( \tau_{21}^{x2y1} + \tau_{21}^{r2y1} \right) \right]; \]

\[ \beta_{1} C_{2z, i\alpha} = -q_{x} C_{2z, i\alpha} + q_{y} C_{2y, i\alpha}; \]

\[ \beta_{1} D_{2z, i\alpha} = -q_{x} D_{2z, i\alpha} + q_{y} D_{2y, i\alpha}; \]

\[ \gamma_{n}(q) = \exp \{ \alpha_{n} q \}; \quad n = 2, 3; \quad \alpha, \gamma = x, y; \]

\[ 1^{2}_{x} = \frac{q_{y} + \gamma_{x}^{2}}{\gamma_{x}^{2}}; \quad 1^{3}_{x} = \frac{q_{y} + \gamma_{x}^{3}}{\gamma_{x}^{3}}; \quad 1^{3}_{x} = \frac{q_{y} + \gamma_{x}^{3}}{\gamma_{x}^{3}}; \]

\[ \gamma_{1} = \gamma_{1} \left( \gamma_{2} \right) \left( \gamma_{3} \right); \quad \gamma_{2} = \gamma_{2} \left( \gamma_{1} \right) \left( \gamma_{3} \right); \]

\[ \gamma_{3} = \gamma_{3} \left( \gamma_{1} \right) \left( \gamma_{2} \right); \]

\[ \gamma_{4} = \gamma_{4} \left( \gamma_{1} \right) \left( \gamma_{2} \right) \left( \gamma_{3} \right); \]
Calculating the properties of heterogeneous mixtures with the help of a phenomenological function

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A method is proposed for calculating the properties of heterogeneous mixtures with the help of a phenomenological function which makes it possible to incorporate the structure of the material, i.e., the spatial distribution of the components. The corresponding function contains two coefficients, one of which determines the type of problem (a site problem or a bond problem), while the second has a power-law relationship with the threshold concentration of inclusions. These expressions can describe the properties of heterogeneous mixtures over their entire range of existence, from a series connection of the components to a parallel one.

Developments in science and technology are making heterogeneous materials important on a progressively broader scale in some diverse fields of technology: electrotechnology, powder metallurgy, heat engineering, etc. The research on the properties of these materials dates back more than 100 years. A treatise by James Clerk Maxwell, one chapter of which was devoted to the properties of heterogeneous mixtures, can be regarded as the first fundamental work in this direction. The further development of this research by Lorentz, Rayleigh, Vicker, Brillgeman, Odelevskii, Landau, and Lifshitz among many others laid the foundation for the theory of heterogeneous mixtures. That theory continues to develop today.

It should be recognized that the methods which have

\[
D_{in}^{(2)} = \frac{\tau_{12}(\beta)}{\beta_n + \beta_2} \left[ a_1 + 2\beta_2 \beta_n - \tau_{12}(\beta) r_{32}(\beta) b_1 U_2(q) \right],
\]

\[
D_{in}^{(1)(\alpha)} = \frac{a_3 \xi_{13}}{\beta_1 + \beta_2} - \frac{q^2 D_{in}^{(1)}}{\Omega},
\]

\[
D_{in}^{(2)(\alpha)} = \tau_{12}(\beta) \frac{a_3 \xi_{13}}{\beta_1 + \beta_2} - \frac{q^2 D_{in}^{(2)}}{\Omega},
\]

\[
C_{in}^{(1)} = \frac{\beta_1 D_{in}^{(1)}}{\beta_2 + \beta_1} - \frac{\beta_2 \Delta a_3 \xi_{13}}{\beta_2 + \beta_1}, \quad C_{in}^{(2)} = \frac{\beta_2 D_{in}^{(2)}}{\beta_2 + \beta_1} + \tau_{12}(\beta) \frac{\beta_2 \Delta b_2}{\beta_2 + \beta_1},
\]

\[
C_{in}^{(1)(\alpha)} = \frac{\beta_1 D_{in}^{(1)}}{\beta_2 + \beta_1} - \frac{\beta_2 \Delta a_3 \xi_{13}}{\beta_2 + \beta_1} - \frac{q^2 C_{in}^{(1)}}{\Omega}, \quad \Omega = a_3 \xi_{13} \xi_{13},
\]

\[
C_{in}^{(2)(\alpha)} = \frac{\beta_2 D_{in}^{(2)}}{\beta_2 + \beta_1} + \tau_{12}(\beta) \frac{\beta_2 \Delta b_2}{\beta_2 + \beta_1}, \quad a_n = q^2 + \beta_2 \beta_n,
\]

\[
r_{in}(\beta) = \frac{\beta_1 - \beta_n}{\beta_1 + \beta_n}, \quad \xi = 1 - \tau_{12}(\beta) r_{32}(\beta) U_2(q),
\]

\[
\xi_{13} = 1 - \tau_{12}(\beta) r_{32}(\beta) \frac{b_3 b_1 U_2(q)}{a_3 a_1}, \quad b_n = q^2 - \beta_2 \beta_n.
\]